Geometry of Lattice Polygons

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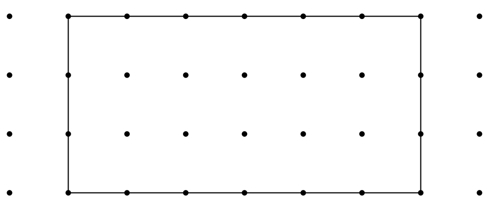
1 Constructing shapes on square lattice

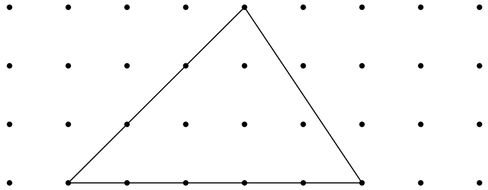
Let us start with a series of warm up questions to make sure that you understand the definition of a lattice polygon. You may draw on the provided dotted paper. The last question is more challenging, so feel free to move on if you have been stuck there for too long.

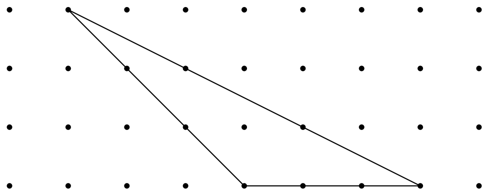
1. Draw a lattice square.
2. Draw a lattice pentagon.
3. Draw a lattice right triangle.
4. (optional) Draw a lattice equilateral triangle.

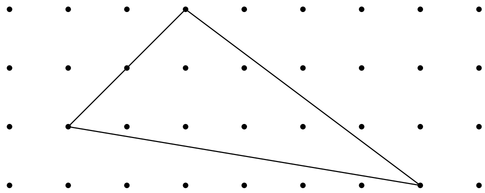
2 Computing areas of lattice polygons

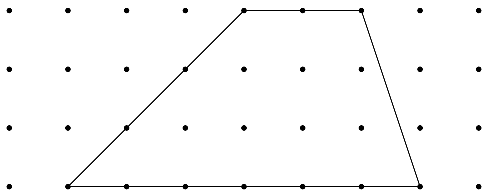
Our goal is to come up with a simple formula for the area of an arbitrary lattice polygon, so it is useful to review methods of computing area which your learned recently. Find the areas of the following lattice polygons.

5. 

6. 

7. 

8. 

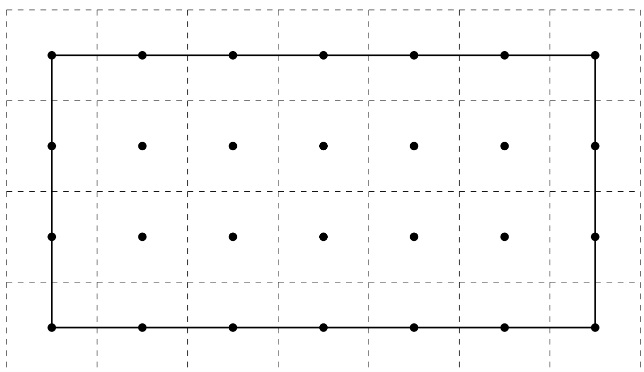
9. 

3 Discovering the formula

We are now ready to discover the magic area formula which works for any lattice polygon, no matter how complicated. The starting point is our intuition that the number of lattice points inside a lattice polygon ought to give a good approximation to the area enclosed by the lattice polygon. We will successively refine that intuition until we arrive at an exact formula. Let us start with the simplest lattice polygon before gradually moving on to more complicated lattice polygons.

3.1 The case of a rectangle

The following is a picture of a lattice rectangle. The dotted lines mark out squares of unit area around each lattice point to help you work out the contribution of each lattice point to the total area enclosed by the rectangle.



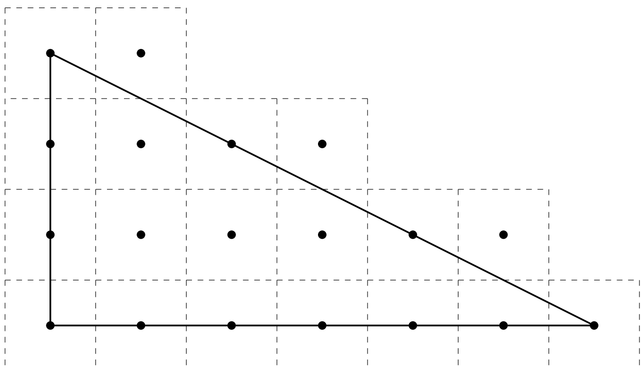
10. What is the area of the region enclosed by the lattice rectangle?
11. Consider lattice points which belong to the interior of the rectangle; in other words, lattice points which are strictly within the rectangle. How much does each such lattice point contribute to the total area enclosed by the rectangle?
12. Consider lattice points which belong to an edge, but not to any vertex, of the rectangle. How much does each such lattice point contribute to the total area enclosed by the rectangle?
13. How much does each lattice point at a vertex of the rectangle contribute to the total area enclosed by the rectangle?

Let I be the number of interior points, E be the number of edge points, and V be the number of vertex points.

14. Based on your previous answers, write down an expression for the area of the rectangle in terms of I, E, and V.
15. Verify that your formula does give the right area for the lattice rectangle above, as well as a few new rectangles you draw.

It appears that you now have a formula which works for all rectangles! You may want to show your formula to a teacher.

3.2 The case of a right triangle



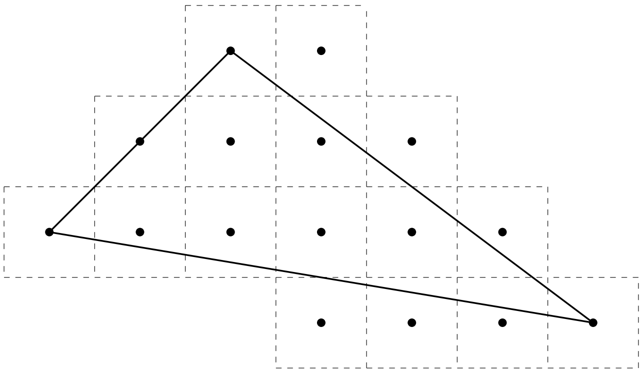
16. How much does each edge lattice point contribute to the total area enclosed by the triangle?
17. How much do vertex lattice points as a whole contribute to the total area enclosed by the triangle? It may help to cut out and rearrange the relevant portions of the squares around the vertex lattice points.

18. How much do the interior lattice points contribute to the total area enclosed by the triangle? Once again, you may want to consider cutting and rearranging.

19. Write down an expression for the area of the triangle in terms of I, E, and V.

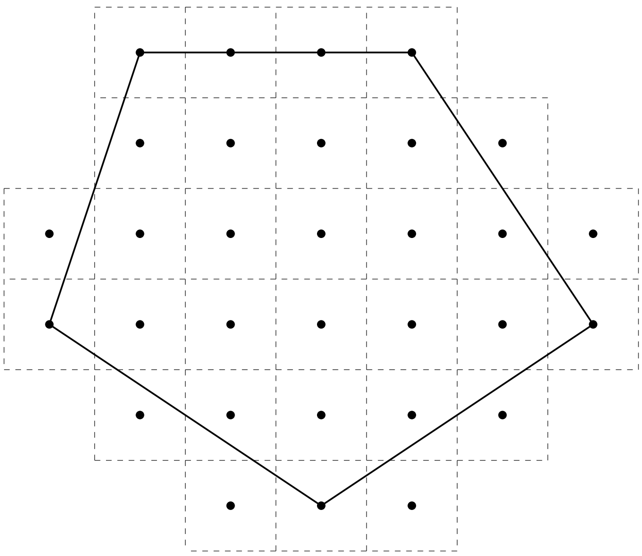
20. Does your formula give the correct area for the triangle above?

3.3 The case of a more general triangle



21. As before, work out the contributions of interior, edge, and vertex lattice points to come up with a formula involving I, E, and V. Verify that your formula does give the right area. Note that you have already computed the area in Section 2.

3.4 The case of a pentagon



22. By this point, you should have an idea of the relationship between the area of a lattice polygon and the numbers I, E, and V. Check that your formula does give the right area for the pentagon above.

4 The magic area formula

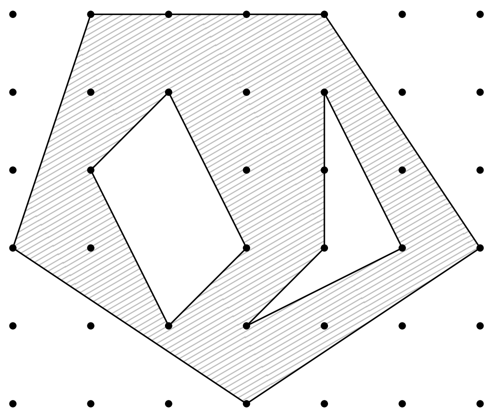
Write down your formula for the area enclosed by a lattice polygon in terms of I, E, and V. This is a good time to show off your formula to a teacher if you have not done so.

Area =

You have done well to discover the formula for the area enclosed by an arbitrary lattice polygon. You are free to move on to the next section, but if you want to go deeper, check out the following suggestions for further exploration.

23. This magic area formula is usually written in terms of I and B, where $B = E+V$ is the number of boundary lattice points, making no distinction between edge and vertex lattice points. Can you translate your formula to the more conventional form known as Pick's Theorem?

24. What modifications do you need to make to your formula for area (or the more conventional one) to make it work for lattice polygons with holes? For example, can we express the area of the shaded region below in terms of the numbers of lattice points inside the shaded region and on its boundary?



25. What we have done in the previous section is to examine a few sample lattice polygons to guess what the formula for the area ought to be. We still have not

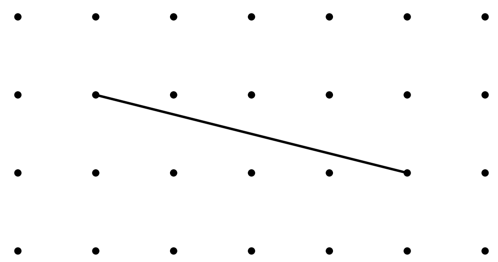
proven that your formula works for every lattice polygons, even ones nobody has ever drawn. One way to go about doing this is to first demonstrate that the formula works for any lattice triangle, and then reduce the case of general lattice polygons to the case of triangles. Feel free to talk to a teacher for hints to get started.

5 Investigating lattice equilateral triangles

If you had some trouble constructing a lattice equilateral triangle in Section 1, you may want to follow the following sequence of problems.

26. If an equilateral triangle has side length s, what is its area in terms of s? The Pythagorean theorem may be useful here.

27. Tsuneo claimed that he successfully constructed a lattice equilateral triangle using the line segment depicted below as one of the edges. Without drawing anything, can you calculate the area of Tsuneo's equilateral triangle?



28. Now think back to your magic formula. What kinds of numbers are I, E, V, and B? What can you conclude about Tsuneo's claim?

29. Can you now construct a lattice equilateral triangle?